Mct/ROB/200 Robotics, Spring Term 12-13

Lecture 6 – Friday March 22, 2013

Differential Motions and Velocities

Objectives

When you have finished this lecture you should be able to:

- Understand the differential kinematics problem of the robot.
- Learn how to derive forward and inverse instantaneous kinematic equations of the robot.
- Understand kinematic singularity.

Outline

- Differential Kinematics
- Differential Motions
- Differential Motions of a Frame
- Forward Instantaneous Kinematics
- Inverse Instantaneous Kinematics
- Kinematic Singularity
- Summary

Outline

- **<u>Differential Kinematics</u>**
- Differential Motions
- Differential Motions of a Frame
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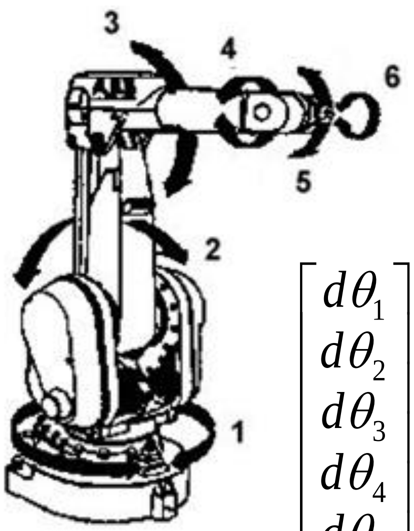
Differential Kinematics



Serial chain manipulator

$$\begin{bmatrix} d\theta_1 \\ d\theta_2 \\ d\theta_3 \\ d\theta_4 \\ d\theta_5 \\ d\theta_6 \end{bmatrix}$$

Given: The positions of all members of the chain and the rates of motion about all the joints.

Required: The total velocity of the end-effector.

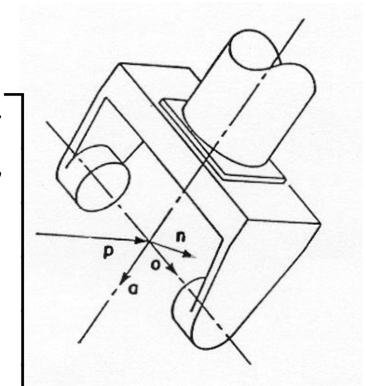
Forward Instantaneous Kinematics →

← Inverse Instantaneous Kinematics

Given: The positions of all members of the chain and the total velocity of the end-effector.

Required: The rates of motion of all joints.

$$\begin{bmatrix} dx \\ dy \\ dz \\ \delta x \\ \delta y \\ \delta z \end{bmatrix}$$



End-effector

- Forward Instantaneous Kinematics=Forward Jacobian=Forward Differential Kinematics
- Inverse Instantaneous Kinematics=Inverse Jacobian=Inverse Differential Kinematics

Differential Kinematics

- The **rate of motion** about the joint is the angular velocity of rotation about a revolute joint or the translational velocity of sliding along a prismatic joint.
- The **total velocity** of a member is the velocity of the origin of the reference frame fixed to it combined with its angular velocity. That is, the total velocity has **six independent components** and therefore, completely represents the velocity field of the member.
- It is important to note that this definition includes an assumption that the **pose of the mechanism is completely known**. In most situations, this means that either the forward or inverse position kinematics problem must be solved before the differential kinematics problem can be addressed. The same is true of the inverse instantaneous kinematics problem.

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Differential Motions

Differential motions are **small movements** of mechanisms (e.g., robots) that can be used to derive velocity relationships between different parts of the mechanism.

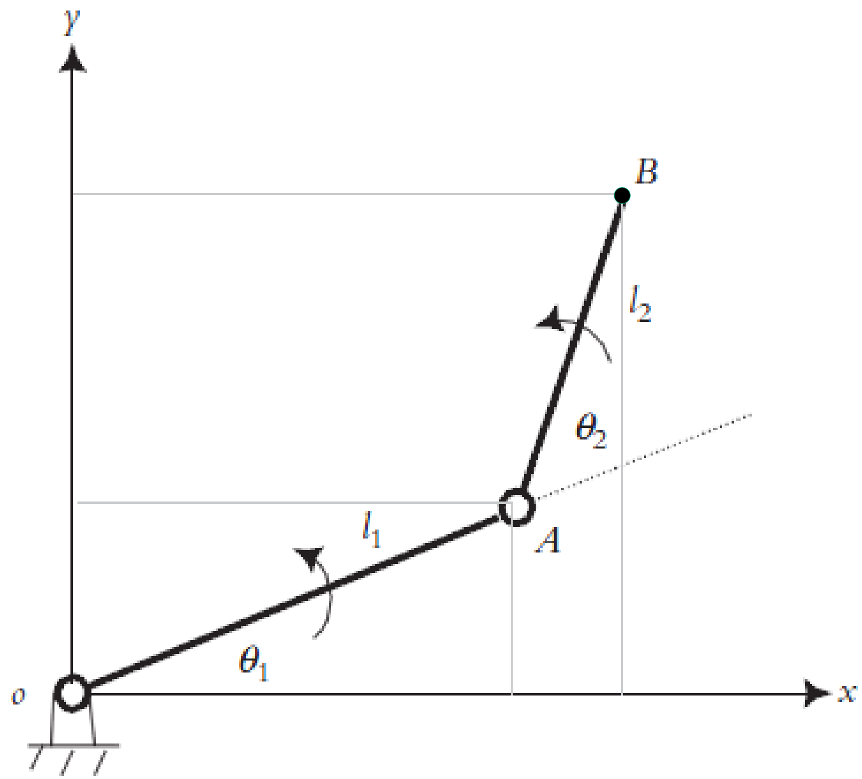
- **2-DOF planar mechanism**

The equations that describe the position of point B are as follows:

$$x_B = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2)$$
$$y_B = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2)$$

Differentiating these equations gives:

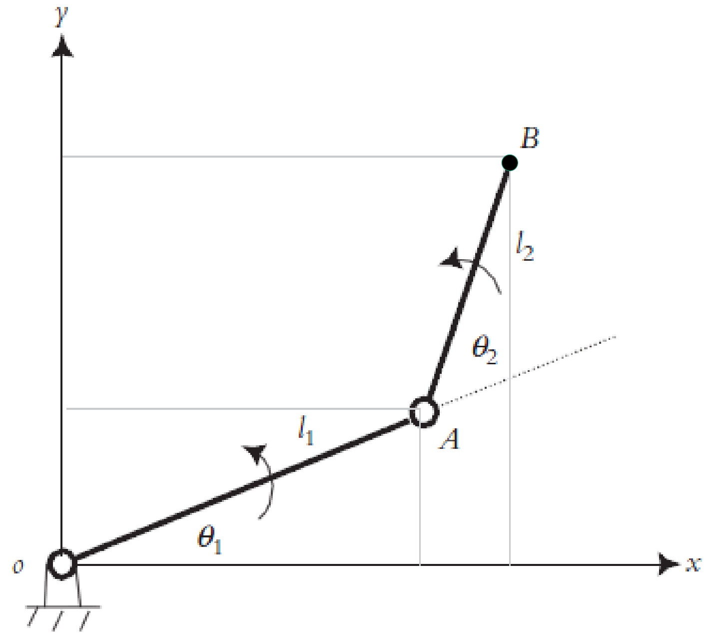
$$dx_B = -l_1 \sin\theta_1 d\theta_1 - l_2 \sin(\theta_1 + \theta_2)(d\theta_1 + d\theta_2)$$
$$dy_B = l_1 \cos\theta_1 d\theta_1 + l_2 \cos(\theta_1 + \theta_2)(d\theta_1 + d\theta_2)$$



2-DOF planar mechanism

Differential Motions

- **2-DOF planar mechanism (cont'd)**



$$dx_B = -l_1 \sin\theta_1 d\theta_1 - l_2 \sin(\theta_1 + \theta_2)(d\theta_1 + d\theta_2)$$
$$dy_B = l_1 \cos\theta_1 d\theta_1 + l_2 \cos(\theta_1 + \theta_2)(d\theta_1 + d\theta_2)$$

and in matrix form:

$$\begin{bmatrix} dx_B \\ dy_B \end{bmatrix} = \begin{bmatrix} -l_1\sin\theta_1 - l_2\sin(\theta_1+\theta_2) & -l_2\sin(\theta_1+\theta_2) \\ l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) & l_2\cos(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} d\theta_1 \\ d\theta_2 \end{bmatrix}$$

Differential motion of B | **Jacobian Matrix** | **Differential motion of joints**

Differential Motions

- **2-DOF planar mechanism (cont'd)**

$$\begin{bmatrix} \mathbf{dx_B} \\ \mathbf{dy_B} \end{bmatrix} = \begin{bmatrix} -\mathbf{l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \theta_2)} & -\mathbf{l_2 \sin(\theta_1 + \theta_2)} \\ \mathbf{l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2)} & \mathbf{l_2 \cos(\theta_1 + \theta_2)} \end{bmatrix} \begin{bmatrix} \mathbf{d\theta_1} \\ \mathbf{d\theta_2} \end{bmatrix}$$

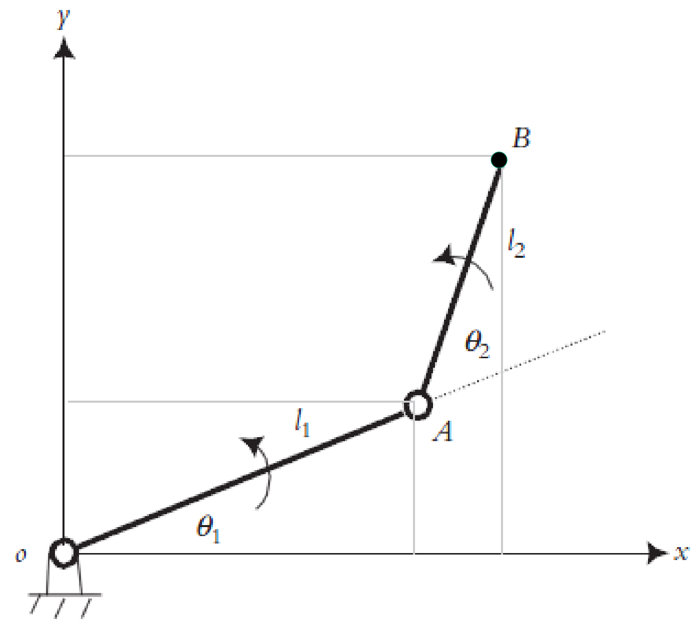
This is the **differential motion relationship**. A differential motion is, by definition, a small movement.

Therefore, if it is measured in—or calculated for—**a small period of time** (a differential or small time), a **velocity relationship** can be found.

$$\begin{bmatrix} \mathbf{dx_B} \\ \mathbf{dy_B} \end{bmatrix} \Big/ dt = \begin{bmatrix} -\mathbf{l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \theta_2)} & -\mathbf{l_2 \sin(\theta_1 + \theta_2)} \\ \mathbf{l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2)} & \mathbf{l_2 \cos(\theta_1 + \theta_2)} \end{bmatrix} \begin{bmatrix} \mathbf{d\theta_1} \\ \mathbf{d\theta_2} \end{bmatrix} \Big/ dt$$

Differential Motions

- **2-DOF planar mechanism (cont'd)**



$$\mathbf{J} = \begin{bmatrix} -l_1\sin\theta_1 - l_2\sin(\theta_1+\theta_2) & -l_2\sin(\theta_1+\theta_2) \\ l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) & l_2\cos(\theta_1+\theta_2) \end{bmatrix}$$

◇ The Jacobian is a **representation of the geometry** of the elements of a mechanism in time.

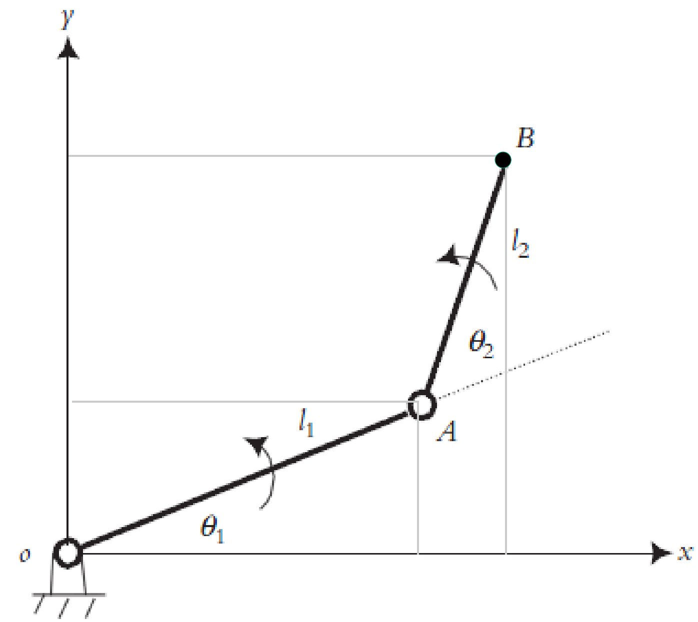
◇ Jacobian is **time-related**; since the values of joint angles vary in time, the magnitude of the elements of the Jacobian vary in time as well.

Differential Motions

- **2-DOF planar mechanism (cont'd)**

$$\begin{bmatrix} dx_B \\ dy_B \end{bmatrix} = J \begin{bmatrix} d\theta_1 \\ d\theta_2 \end{bmatrix} \quad \text{or}$$

$$J = \frac{\begin{bmatrix} dx_B \\ dy_B \end{bmatrix}}{\begin{bmatrix} d\theta_1 \\ d\theta_2 \end{bmatrix}}$$



◇ Jacobian defines how the **end effector changes** relative to **instantaneous changes** in the system.

◇ It allows the conversion of differential motions or velocities of individual joints to differential motions or velocities of points of interest (e.g., the end effector). It also relates the individual joint motions to overall mechanism motions.

Outline

- Differential Kinematics
- Differential Motions
- **Differential Motions of a Frame**
- Forward Instantaneous Kinematics
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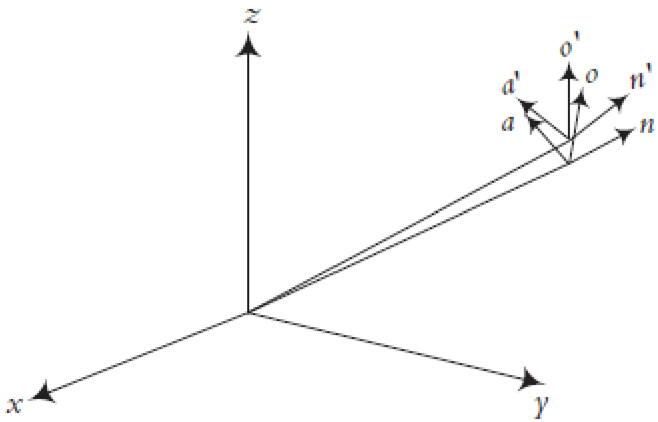
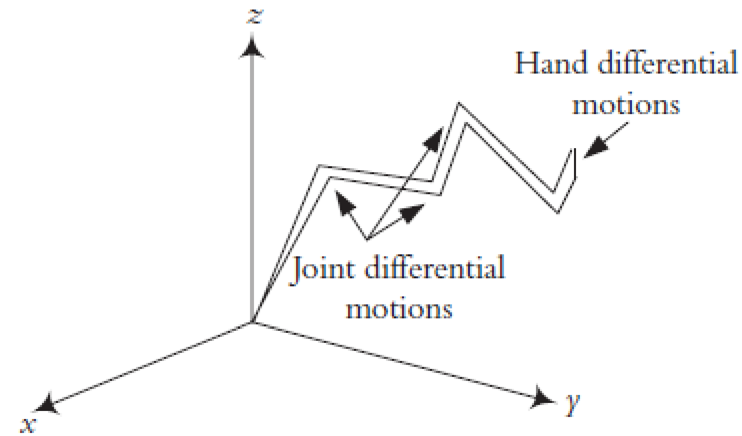
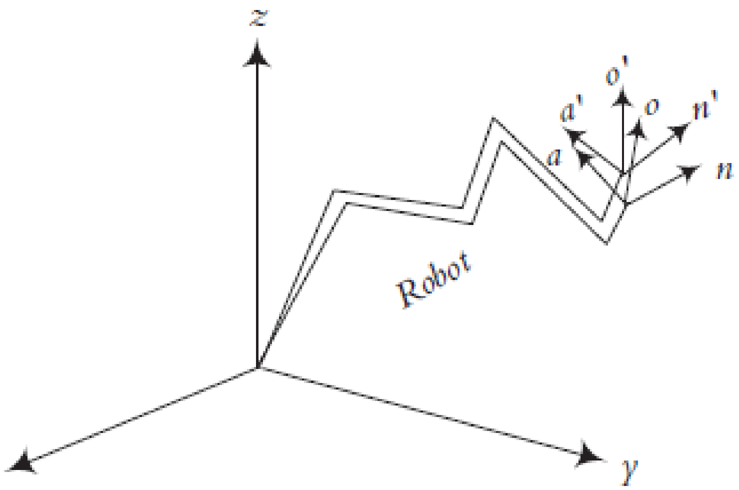
Differential Motions of a Frame



(a) Differential motions of a frame



(b) differential motions of the robot joints and the endplate



(c) differential motions of a frame caused by the differential motions of a robot

Differential Motions of a Frame

Differential motions of a frame can be divided into the following:

- Differential translations;
- Differential rotations;
- Differential transformations (combinations of translations and rotations).

Differential Motions of a Frame

- **Differential Translations**

A differential translation is the translation of a frame at differential values.

Therefore, it can be represented by ***Trans(dx, dy, dz)***.

This means the frame has moved a differential amount along the x-, y-, and z-axes.

Differential Motions of a Frame

- **Differential Translations**

Example: A frame B has translated a differential amount of Trans(0.01, 0.05, 0.03) units. Find its new location and orientation.

$$B = \begin{bmatrix} 0.707 & 0 & -0.707 & 5 \\ 0 & 1 & 0 & 4 \\ 0.707 & 0 & 0.707 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution: Since the differential motion is only a translation, the orientation of the frame should not be affected. The new location of the frame is:

$$B = \begin{bmatrix} 1 & 0 & 0 & 0.01 \\ 0 & 1 & 0 & 0.05 \\ 0 & 0 & 1 & 0.03 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0.707 & 0 & -0.707 & 5 \\ 0 & 1 & 0 & 4 \\ 0.707 & 0 & 0.707 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.707 & 0 & -0.707 & 5.01 \\ 0 & 1 & 0 & 4.05 \\ 0.707 & 0 & 0.707 & 9.03 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Differential Motions of a Frame

- **Differential Rotations about the Reference Axes**

A differential rotation is a small rotation of the frame. It is generally represented by **Rot(q,dθ)**, which means that the frame has rotated an angle of dθ about an axis q.

Specifically, differential rotations about the x-, y-, and z-axes are defined by **δx, δy, and δz**.

Since the rotations are **small amounts**, we can use the following **approximations**:

$$\sin \delta x = \delta x \text{ (in radians)}$$
$$\cos \delta x = 1$$

Differential Motions of a Frame

- **Differential Rotations about the Reference Axes**

Then, the rotation matrices representing differential rotations about the x-, y-, and z-axes will be:

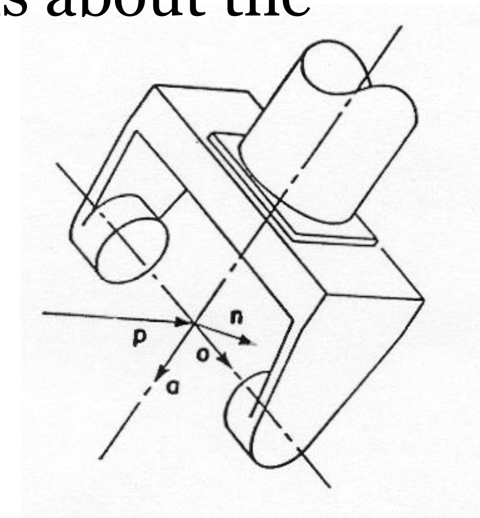
$$Rot(x,\delta x)=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\delta x & -\sin\delta x & 0 \\ 0 & \sin\delta x & \cos\delta x & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\delta x & 0 \\ 0 & \delta x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Rot(y,\delta y)=\begin{bmatrix} 1 & 0 & \delta y & 0 \\ 0 & 1 & 0 & 0 \\ -\delta y & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad Rot(z,\delta z)=\begin{bmatrix} 1 & -\delta z & 0 & 0 \\ \delta z & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Differential Motions of a Frame

- **Differential Rotations about the Reference Axes**

Similarly, we can also define differential rotations about the current axes as:



$$Rot(n,\delta n)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & -\delta n & 0\\ 0 & \delta n & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix},$$

$$Rot(o,\delta o)=\begin{bmatrix}1 & 0 & \delta o & 0\\ 0 & 1 & 0 & 0\\ -\delta o & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}, \quad Rot(a,\delta a)=\begin{bmatrix}1 & -\delta a & 0 & 0\\ \delta a & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Differential Motions of a Frame

- **Differential Rotations about the Reference Axes**

Order of multiplication in Successive Rotations:

$$Rot(x,\delta x)Rot(y,\delta y)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & -\delta x & 0\\ 0 & \delta x & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & \delta y & 0\\ 0 & 1 & 0 & 0\\ -\delta y & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & \delta y & 0\\ \delta x\delta y & 1 & -\delta x & 0\\ -\delta y & \delta x & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$Rot(y,\delta y)Rot(x,\delta x)=\begin{bmatrix}1 & 0 & \delta y & 0\\ 0 & 1 & 0 & 0\\ -\delta y & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & -\delta x & 0\\ 0 & \delta x & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & \delta x\delta y & \delta y & 0\\ 0 & 1 & -\delta x & 0\\ -\delta y & \delta x & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

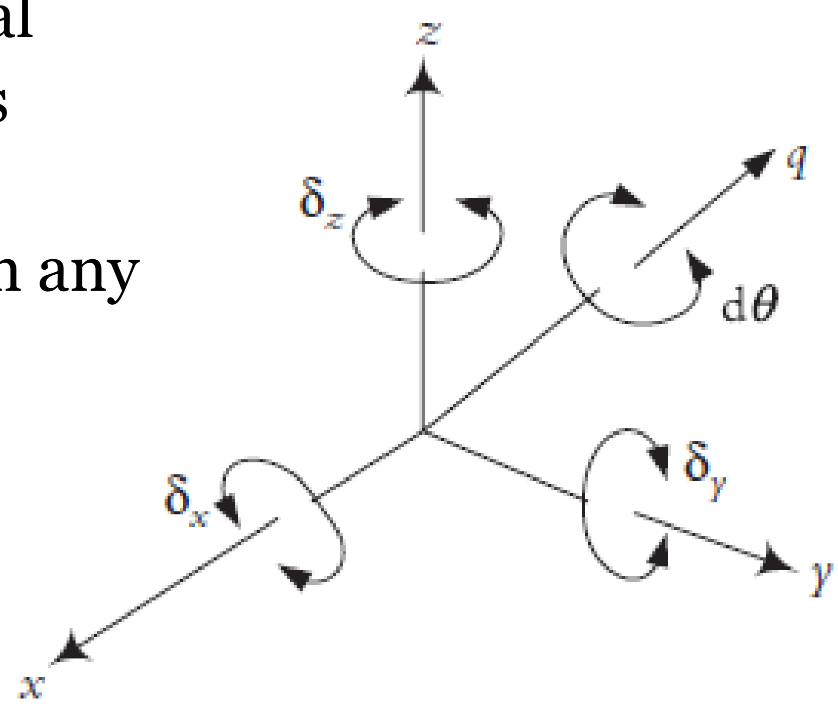
If we set **higher-order differentials** such as **δx δy** to zero, the results are exactly the same. Consequently, for differential motions, the order of multiplication is **no longer important** and Rot(x, δx)Rot(y, δy)= Rot(y, δy)Rot(x, δx).

Differential Motions of a Frame

- **Differential Rotation about a General Axis q**

We can assume that a differential rotation about a general axis q is composed of three differential rotations about the three axes, in any order, or

$$(d\theta)q = (\delta x)i + (\delta y)\,j + (\delta z)k$$



Differential Motions of a Frame

- **Differential Rotation about a General Axis q**

$$Rot(q, d\theta) = Rot(x, \delta x)Rot(y, \delta y)Rot(z, \delta z)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\delta x & 0 \\ 0 & \delta x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \delta y & 0 \\ 0 & 1 & 0 & 0 \\ -\delta y & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\delta z & 0 & 0 \\ \delta z & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\delta z & \delta y & 0 \\ \delta x\delta y + \delta z & -\delta x\delta y\delta z + 1 & -\delta x & 0 \\ -\delta y + \delta x\delta z & \delta x + \delta y\delta z & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If we neglect all higher-order differentials, we get:

$$Rot(q, d\theta) = Rot(x, \delta x)Rot(y, \delta y)Rot(z, \delta z) = \begin{bmatrix} 1 & -\delta z & \delta y & 0 \\ \delta z & 1 & -\delta x & 0 \\ -\delta y & \delta x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Differential Motions of a Frame

- **Differential Rotation about a General Axis q**

Example: Find the total differential transformation caused by small rotations about the three axes of δx=0.1, δy= 0.05, δz=0.02 radians.

Solution:

$$Rot(q, d\theta) = \begin{bmatrix} 1 & -\delta z & \delta y & 0 \\ \delta z & 1 & -\delta x & 0 \\ -\delta y & \delta x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -0.02 & 0.05 & 0 \\ 0.02 & 1 & -0.1 & 0 \\ -0.05 & 0.1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Differential Motions of a Frame

- **Differential Transformations of a Frame**

The differential transformation of a frame is a **combination of differential translations and rotations** in any order.

If we denote:

T: the original frame

dT: the change in the frame T as a result of a differential transformation, then:

$$[\mathbf{T} + \mathbf{dT}] = [\mathbf{Trans}(\mathbf{dx}, \mathbf{dy}, \mathbf{dz})\mathbf{Rot}(\mathbf{q}, \mathbf{d\theta})][\mathbf{T}]$$

or

$$[\mathbf{dT}] = [\mathbf{Trans}(\mathbf{dx}, \mathbf{dy}, \mathbf{dz})\mathbf{Rot}(\mathbf{q}, \mathbf{d\theta}) - \mathbf{I}][\mathbf{T}]$$

where **I** is a **unit matrix**.

Differential Motions of a Frame

- **Differential Transformations of a Frame**

$$[dT] = [Trans(dx, dy, dz)Rot(q, d\theta) - I][T]$$

The above equation can be written as:

$$[dT] = [\Delta][T]$$

where

$$\Delta = [Trans(dx, dy, dz)Rot(q, d\theta) - I]$$

[Δ] (or simply Δ) is called **differential operator**.

Differential Motions of a Frame

- **Differential Transformations of a Frame**

$$\Delta = Trans(dx, dy, dz) \times Rot(q, d\theta) - I$$

$$= \begin{bmatrix} 1 & 0 & 0 & dx \\ 0 & 1 & 0 & dy \\ 0 & 0 & 1 & dz \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\delta z & \delta y & 0 \\ \delta z & 1 & -\delta x & 0 \\ -\delta y & \delta x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Delta = \begin{bmatrix} 0 & -\delta z & \delta y & dx \\ \delta z & 0 & -\delta x & dy \\ -\delta y & \delta x & 0 & dz \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

As you can see, the differential operator is not a transformation matrix, or a frame. It does not follow the required format either; it is only an operator, and it **yields the changes in a frame**.

Differential Motions of a Frame

- **Differential Transformations of a Frame**

Example: Find the effect of a differential rotation of 0.1 rad about the y-axis followed by a differential translation of [0.1, 0, 0.2] on the given frame B.

$$B = \begin{bmatrix} 0 & 0 & 1 & 10 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution:

$$\text{For } dx = 0.1, \quad dy = 0, \quad dz = 0.2, \quad \delta x = 0, \quad \delta y = 0.1, \quad \delta z = 0$$

$$[dB] = [\Delta][B] = \begin{bmatrix} 0 & 0 & 0.1 & 0.1 \\ 0 & 0 & 0 & 0 \\ -0.1 & 0 & 0 & 0.2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 10 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0.1 & 0 & 0.4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1 & -0.8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Differential Motions of a Frame

- **Differential Transformations of a Frame**

Example: Find the location and the orientation of frame B after the move.

$$B = \begin{bmatrix} 0 & 0 & 1 & 10 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad [dB] = \begin{bmatrix} 0 & 0.1 & 0 & 0.4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1 & -0.8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution: The new location and orientation of the frame can be found by adding the changes to the original values. The result is:

$$B_{new} = B_{original} + dB$$

$$= \begin{bmatrix} 0 & 0 & 1 & 10 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0.1 & 0 & 0.4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1 & -0.8 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0.1 & 1 & 10.4 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & -0.1 & 2.2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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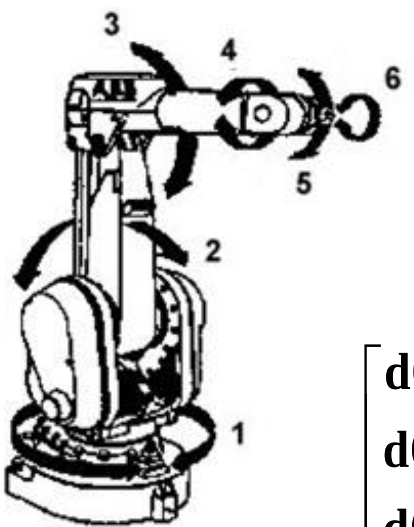
Forward Instantaneous Kinematics

Given: The positions of all members of the chain and the rates of motion about all the joints.

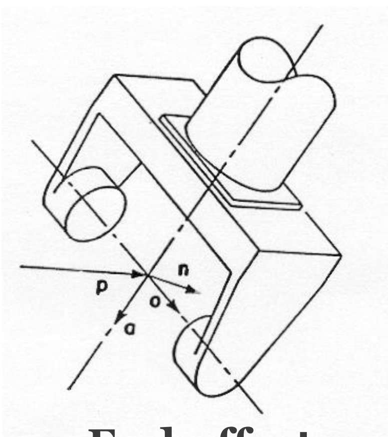
Required: The total velocity of the end-effector.



Serial chain manipulator

Forward Instantaneous Kinematics →

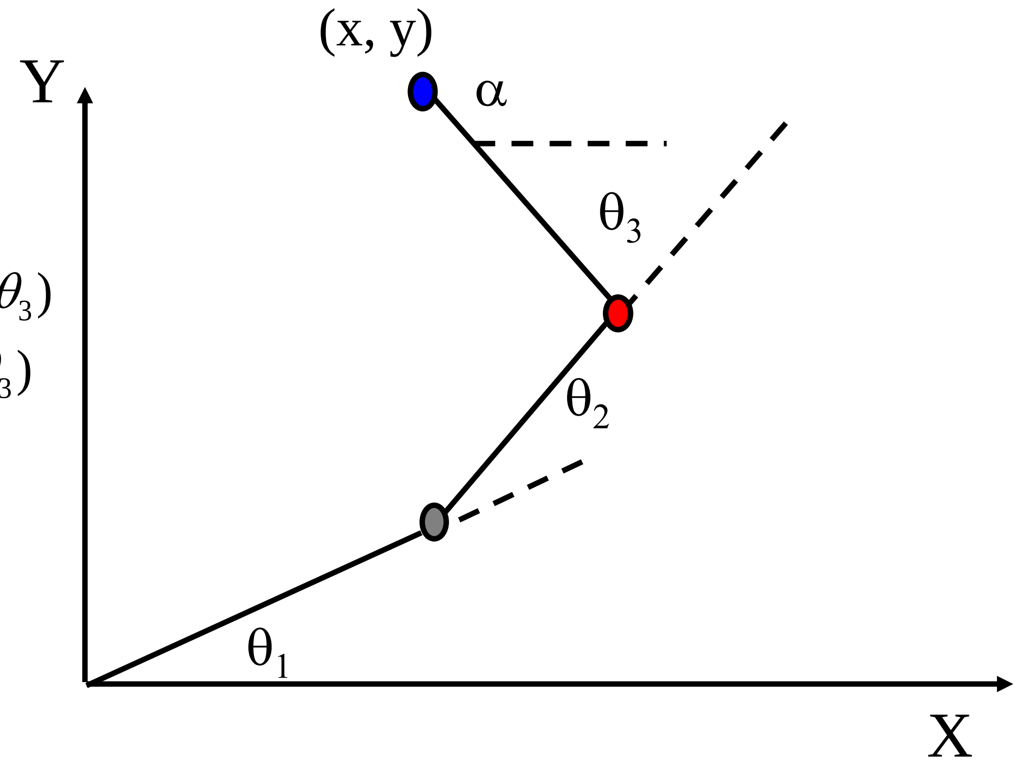
$$\begin{bmatrix} d\theta_1 \\ d\theta_2 \\ d\theta_3 \\ d\theta_4 \\ d\theta_5 \\ d\theta_6 \end{bmatrix} \rightarrow \begin{bmatrix} dx \\ dy \\ dz \\ \delta x \\ \delta y \\ \delta z \end{bmatrix}$$



End-effector

Forward Instantaneous Kinematics

- **3-DOF Plan Robot**



Forward Kinematics Solution:

$$x = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) + l_3 \cos(\theta_1 + \theta_2 + \theta_3)$$
$$y = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) + l_3 \sin(\theta_1 + \theta_2 + \theta_3)$$
$$\alpha = \theta_1 + \theta_2 + \theta_3$$

Differentiating these equations give:

$$v_x = -(l_1 S_1 + l_2 S_{12} + l_3 S_{123})\omega_1 - (l_2 S_{12} + l_3 S_{123})\omega_2 - l_3 S_{123}\omega_3$$
$$v_y = (l_1 C_1 + l_2 C_{12} + l_3 C_{123})\omega_1 + (l_2 C_{12} + l_3 C_{123})\omega_2 + l_3 C_{123}\omega_3$$
$$\omega_\alpha = \omega_1 + \omega_2 + \omega_3$$

Forward Instantaneous Kinematics

- **3-DOF Plan Robot**

In matrix form:

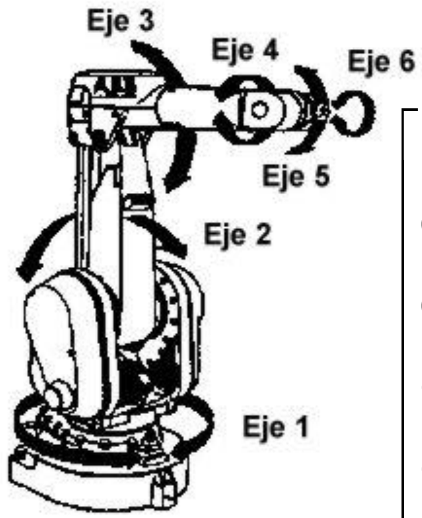
$$\begin{bmatrix} v_x \\ v_y \\ \omega_\alpha \end{bmatrix} = \begin{bmatrix} -(l_1 S_1 + l_2 S_{12} + l_3 S_{123}) & -(l_2 S_{12} + l_3 S_{123}) & -l_3 S_{123} \\ (l_1 C_1 + l_2 C_{12} + l_3 C_{123}) & (l_2 C_{12} + l_3 C_{123}) & l_3 C_{123} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

Jacobian Matrix:

$$J = \begin{bmatrix} -(l_1 S_1 + l_2 S_{12} + l_3 S_{123}) & -(l_2 S_{12} + l_3 S_{123}) & -l_3 S_{123} \\ (l_1 C_1 + l_2 C_{12} + l_3 C_{123}) & (l_2 C_{12} + l_3 C_{123}) & l_3 C_{123} \\ 1 & 1 & 1 \end{bmatrix}$$

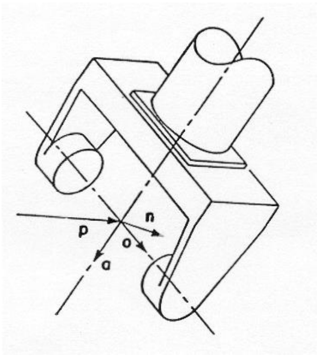
Forward Instantaneous Kinematics

- **Calculation of Jacobian**



Serial chain manipulator

$$\begin{bmatrix} d\theta_1 \\ d\theta_2 \\ d\theta_3 \\ d\theta_4 \\ d\theta_5 \\ d\theta_6 \end{bmatrix} \xrightarrow{\text{Forward Instantaneous Kinematics}} \begin{bmatrix} dx \\ dy \\ dz \\ \delta x \\ \delta y \\ \delta z \end{bmatrix}$$



End-effector

$$\begin{bmatrix} dx \\ dy \\ dz \\ \delta x \\ \delta y \\ \delta z \end{bmatrix} = \begin{bmatrix} \text{Robot} \\ \text{Jacobian} \\ J \end{bmatrix} \begin{bmatrix} d\theta_1 \\ d\theta_2 \\ d\theta_3 \\ d\theta_4 \\ d\theta_5 \\ d\theta_6 \end{bmatrix}$$

Forward Instantaneous Kinematics

- **Calculation of Jacobian**

  Assume that the forward kinematics solution is as follows:

$$x = f_x(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6)$$

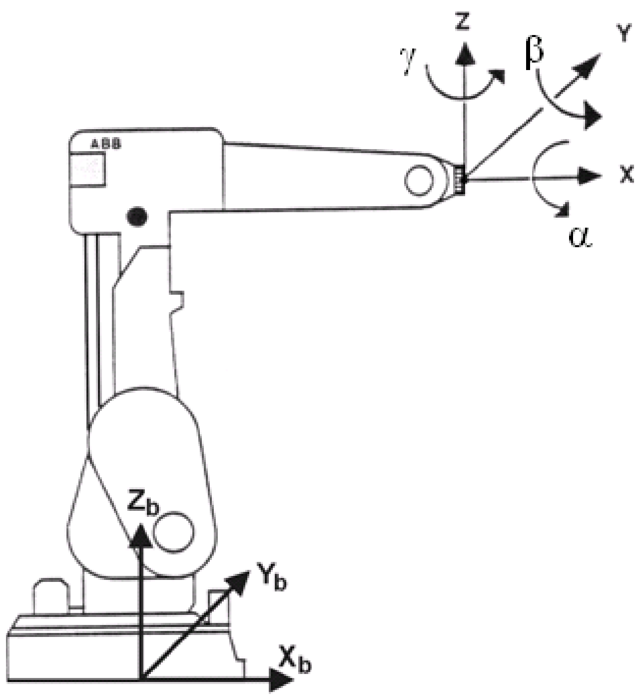
$$y = f_y(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6)$$

$$z = f_x(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6)$$

$$\alpha = f_\alpha(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6)$$

$$\beta = f_\beta(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6)$$

$$\lambda = f_\gamma(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6)$$



Forward Instantaneous Kinematics

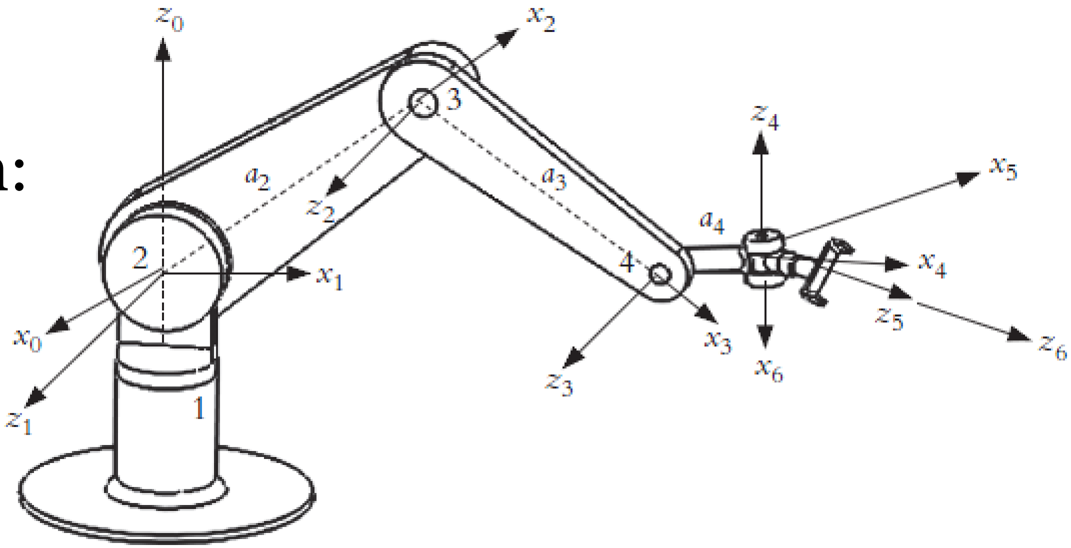
- **Calculation of Jacobian**

$$\mathbf{J}=\begin{bmatrix} J_{11} & J_{12} & \cdot & \cdot & \cdot & J_{16} \\ J_{21} & J_{21} & \cdot & \cdot & \cdot & J_{26} \\ J_{31} & \cdot & \cdot & \cdot & \cdot & J_{36} \\ J_{41} & \cdot & \cdot & \cdot & \cdot & J_{46} \\ J_{51} & \cdot & \cdot & \cdot & \cdot & J_{56} \\ J_{61} & \cdot & \cdot & \cdot & \cdot & J_{66} \end{bmatrix}=\begin{bmatrix} \frac{\partial f_x}{\partial \theta_1} & \frac{\partial f_x}{\partial \theta_2} & \cdot & \cdot & \cdot & \frac{\partial f_x}{\partial \theta_6} \\ \frac{\partial f_y}{\partial \theta_1} & \cdot & \cdot & \cdot & \cdot & \frac{\partial f_y}{\partial \theta_6} \\ \frac{\partial f_z}{\partial \theta_1} & \cdot & \cdot & \cdot & \cdot & \frac{\partial f_z}{\partial \theta_6} \\ \frac{\partial f_\alpha}{\partial \theta_1} & \cdot & \cdot & \cdot & \cdot & \frac{\partial f_\alpha}{\partial \theta_6} \\ \frac{\partial f_\beta}{\partial \theta_1} & \cdot & \cdot & \cdot & \cdot & \frac{\partial f_\beta}{\partial \theta_6} \\ \frac{\partial f_\gamma}{\partial \theta_1} & \cdot & \cdot & \cdot & \cdot & \frac{\partial f_\gamma}{\partial \theta_6} \end{bmatrix}$$

Forward Instantaneous Kinematics

- **Calculation of Jacobian**

Forward Kinematics Solution:



#	θ	d	a	α
0–1	θ_1	0	0	90
1–2	θ_2	0	a_2	0
2–3	θ_3	0	a_3	0
3–4	θ_4	0	a_4	−90
4–5	θ_5	0	0	90
5–6	θ_6	0	0	0

$$^{R}T_{H} = A_1 A_2 A_3 A_4 A_5 A_6$$

$$= \begin{bmatrix} C_1(C_{234}C_5C_6 - S_{234}S_6) - S_1S_5C_6 & C_1(-C_{234}C_5C_6 - S_{234}C_6) + S_1S_5S_6 & C_1(C_{234}S_5) + S_1C_5 & C_1(C_{234}a_4 + C_{23}a_3 + C_2a_2) \\ S_1(C_{234}C_5C_6 - S_{234}S_6) + C_1S_5C_6 & S_1(-C_{234}C_5C_6 - S_{234}C_6) - C_1S_5S_6 & S_1(C_{234}S_5) - C_1C_5 & S_1(C_{234}a_4 + C_{23}a_3 + C_2a_2) \\ S_{234}C_5C_6 + C_{234}S_6 & -S_{234}C_5C_6 + C_{234}C_6 & S_{234}S_5 & S_{234}a_4 + S_{23}a_3 + S_2a_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Forward Instantaneous Kinematics

- **Calculation of Jacobian**

Consider last column of the forward kinematic equation of the robot is:

$$^{R}T_{H} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} C_1(C_{234}a_4 + C_{23}a_3 + C_2a_2) \\ S_1(C_{234}a_4 + C_{23}a_3 + C_2a_2) \\ S_{234}a_4 + S_{23}a_3 + S_2a_2 \\ 1 \end{bmatrix}$$

Taking the derivative of p_x will yield:

Forward Instantaneous Kinematics

- **Calculation of Jacobian**

$$p_x = C_1(C_{234}a_4 + C_{23}a_3 + C_2a_2)$$

$$dp_x = \frac{\partial p_x}{\partial \theta_1} d\theta_1 + \frac{\partial p_x}{\partial \theta_2} d\theta_2 + \cdots + \frac{\partial p_x}{\partial \theta_6} d\theta_6$$

$$dp_x = -S_1[C_{234}a_4 + C_{23}a_3 + C_2a_2]d\theta_1 + C_1[-S_{234}a_4 - S_{23}a_3 - S_2a_2]d\theta_2$$
$$+ C_1[-S_{234}a_4 - S_{23}a_3]d\theta_3 + C_1[-S_{234}a_4]d\theta_4$$

From this, we can write the first row of the Jacobian as:

Forward Instantaneous Kinematics

- **Calculation of Jacobian**

$$dp_x = -S_1[C_{234}a_4 + C_{23}a_3 + C_2a_2]d\theta_1 + C_1[-S_{234}a_4 - S_{23}a_3 - S_2a_2]d\theta_2$$
$$+C_1[-S_{234}a_4 - S_{23}a_3]d\theta_3 + C_1[-S_{234}a_4]d\theta_4$$

$$\frac{\partial p_x}{\partial \theta_1} = J_{11} = -S_1[C_{234}a_4 + C_{23}a_3 + C_2a_2]$$

$$\frac{\partial p_x}{\partial \theta_2} = J_{12} = C_1[-S_{234}a_4 - S_{23}a_3 - S_2a_2]$$

$$\frac{\partial p_x}{\partial \theta_3} = J_{13} = C_1[-S_{234}a_4 - S_{23}a_3]$$

$$\frac{\partial p_x}{\partial \theta_4} = J_{14} = C_1[-S_{234}a_4]$$

$$\frac{\partial p_x}{\partial \theta_5} = J_{15} = 0$$

$$\frac{\partial p_x}{\partial \theta_6} = J_{16} = 0$$

Forward Instantaneous Kinematics

- **Calculation of Jacobian**

For the second row of the Jacobian, we will have to differentiate the p_y expression of the following equation

$$\begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} C_1(C_{234}a_4 + C_{23}a_3 + C_2a_2) \\ S_1(C_{234}a_4 + C_{23}a_3 + C_2a_2) \\ S_{234}a_4 + S_{23}a_3 + S_2a_2 \\ 1 \end{bmatrix}$$

$$p_y = S_1(C_{234}a_4 + C_{23}a_3 + C_2a_2)$$

$$dp_y = \frac{\partial p_y}{\partial \theta_1} d\theta_1 + \frac{\partial p_y}{\partial \theta_2} d\theta_2 + \cdots + \frac{\partial p_y}{\partial \theta_6} d\theta_6$$

$$dp_y = C_1(C_{234}a_4 + C_{23}a_3 + C_2a_2)d\theta_1 + S_1[-S_{234}a_4(d\theta_2 + d\theta_3 + d\theta_4) - S_{23}a_3(d\theta_2 + d\theta_3) - S_2a_2(d\theta_2)]$$

Forward Instantaneous Kinematics

- **Calculation of Jacobian**

Rearranging the terms will yield:

$$\frac{\partial p_y}{\partial \theta_1} d\theta_1 = J_{21} d\theta_1 = C_1 (C_{234} a_4 + C_{23} a_3 + C_2 a_2) d\theta_1$$

$$\frac{\partial p_y}{\partial \theta_2} d\theta_2 = J_{22} d\theta_2 = S_1 (-S_{234} a_4 - S_{23} a_3 - S_2 a_2) d\theta_2$$

$$\frac{\partial p_y}{\partial \theta_3} d\theta_3 = J_{23} d\theta_3 = S_1 (-S_{234} a_4 - S_{23} a_3) d\theta_3$$

$$\frac{\partial p_y}{\partial \theta_4} d\theta_4 = J_{24} d\theta_4 = S_1 (-S_{234} a_4) d\theta_4$$

$$\frac{\partial p_y}{\partial \theta_5} d\theta_5 = J_{25} d\theta_5 = 0 \quad \text{and} \quad \frac{\partial p_y}{\partial \theta_6} d\theta_6 = J_{26} d\theta_6 = 0$$

The same can be done for the other rows…

Forward Instantaneous Kinematics

- **Calculation of Jacobian: Another Approach**

  In reality, it is actually **a lot simpler** to calculate the Jacobian relative to T_6, the last frame, than it is to calculate it relative to the first frame[1]. Therefore, we will instead use the following approach.

$$\begin{bmatrix} dx \\ dy \\ dz \\ \delta x \\ \delta y \\ \delta z \end{bmatrix} = \begin{bmatrix} \text{Robot} \\ \text{Jacobian} \end{bmatrix} \begin{bmatrix} d\theta_1 \\ d\theta_2 \\ d\theta_3 \\ d\theta_4 \\ d\theta_5 \\ d\theta_6 \end{bmatrix}$$

$$[D] = [J][D_\theta]$$

[1]R. Pual. *Robot Manipulator, Mathematics, Programming, and Control*. The MIT press, 1981.

Forward Instantaneous Kinematics

- **Calculation of Jacobian**

$$[D] = [J][D_\theta]$$

We can write the velocity equation relative to the last frame as:

$$[{}^{T_6}D] = [{}^{T_6}J][D_\theta]$$

This means that for the same joint differential motions, pre-multiplied with the Jacobian relative to the last frame, we get the **hand differential motions relative to the last frame**.

Forward Instantaneous Kinematics

- **Calculation of Jacobian**

How to calculate the Jacobian with respect to the last frame? $\left[{}^{T_6}J\right]$

◇ The differential motion relationship of $[{}^{T_6}\mathbf{D}] = [{}^{T_6}\mathbf{J}][\mathbf{D}_\theta]$ can be written as:

$$\begin{bmatrix} {}^{T_6}d_x \\ {}^{T_6}d_y \\ {}^{T_6}d_z \\ {}^{T_6}\delta_x \\ {}^{T_6}\delta_y \\ {}^{T_6}\delta_z \end{bmatrix} = \begin{bmatrix} {}^{T_6}J_{11} & {}^{T_6}J_{12} & . & . & . & {}^{T_6}J_{16} \\ {}^{T_6}J_{21} & {}^{T_6}J_{22} & . & . & . & {}^{T_6}J_{26} \\ {}^{T_6}J_{31} & . & . & . & . & {}^{T_6}J_{36} \\ {}^{T_6}J_{41} & . & . & . & . & {}^{T_6}J_{46} \\ {}^{T_6}J_{51} & . & . & . & . & {}^{T_6}J_{56} \\ {}^{T_6}J_{61} & . & . & . & . & {}^{T_6}J_{66} \end{bmatrix} \begin{bmatrix} d\theta_1 \\ d\theta_2 \\ . \\ . \\ . \\ d\theta_6 \end{bmatrix}$$

◇ Assuming that any combination of $A_1A_2 \ldots A_n$ can be expressed with a corresponding **n, o, a, p** matrix, the corresponding elements of the matrix will be used to calculate the Jacobian.

Forward Instantaneous Kinematics

- **Calculation of Jacobian**
  - ◇ If joint i under consideration is a **revolute joint**, then:

$$^{T_6}J_{1i} = (-n_x p_y + n_y p_x) \quad ^{T_6}J_{2i} = (-o_x p_y + o_y p_x) \quad ^{T_6}J_{3i} = (-a_x p_y + a_y p_x)$$
$$^{T_6}J_{4i} = n_z \quad ^{T_6}J_{5i} = o_z \quad ^{T_6}J_{6i} = a_z$$

  - ◇ If joint i under consideration is a **prismatic joint**, then:

$$^{T_6}J_{1i} = n_z \quad ^{T_6}J_{2i} = o_z \quad ^{T_6}J_{3i} = a_z$$
$$^{T_6}J_{4i} = 0 \quad ^{T_6}J_{5i} = 0 \quad ^{T_6}J_{6i} = 0$$

  - ◇ For the previous equations, for column i use $^{i-1}T_6$, meaning:

For column 1, use $^{o}T_6 = A_1A_2A_3A_4A_5A_6$
For column 2, use $^{1}T_6 = A_2A_3A_4A_5A_6$
For column 3, use $^{2}T_6 = A_3A_4A_5A_6$
For column 4, use $^{3}T_6 = A_4A_5A_6$
For column 5, use $^{4}T_6 = A_5A_6$
For column 6, use $^{5}T_6 = A_6$

Forward Instantaneous Kinematics

- **Calculation of Jacobian**

Example: Find the ${}^{T6}J_{11}$ and ${}^{T6}J_{41}$ elements of the Jacobian for the simple revolute robot.

Solution: To calculate two elements of the first column of the Jacobian, we need to use $A_1A_2 \ldots A_6$ matrix

$$^{R}T_{\mathrm{H}} = A_1A_2A_3A_4A_5A_6$$

Forward Instantaneous Kinematics

- **Calculation of Jacobian**

Example (cont'd):

$$ {}^{R}T_{\mathrm{H}} = A_1A_2A_3A_4A_5A_6 = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ = \begin{bmatrix} \begin{matrix} C_1(C_{234}C_5C_6 - S_{234}S_6) \\ -S_1S_5C_6 \end{matrix} & \begin{matrix} C_1\begin{pmatrix} -C_{234}C_5C_6 \\ -S_{234}C_6 \end{pmatrix} \\ +S_1S_5S_6 \end{matrix} & \begin{matrix} C_1(C_{234}S_5) \\ +S_1C_5 \end{matrix} & C_1\begin{pmatrix} C_{234}a_4 + C_{23}a_3 \\ +C_2a_2 \end{pmatrix} \\ \begin{matrix} S_1(C_{234}C_5C_6 - S_{234}S_6) \\ +C_1S_5C_6 \end{matrix} & \begin{matrix} S_1\begin{pmatrix} -C_{234}C_5C_6 \\ -S_{234}C_6 \end{pmatrix} \\ -C_1S_5S_6 \end{matrix} & \begin{matrix} S_1(C_{234}S_5) \\ -C_1C_5 \end{matrix} & S_1\begin{pmatrix} C_{234}a_4 + C_{23}a_3 \\ +C_2a_2 \end{pmatrix} \\ S_{234}C_5C_6 + C_{234}S_6 & -S_{234}C_5C_6 + C_{234}C_6 & S_{234}S_5 & \begin{matrix} S_{234}a_4 + S_{23}a_3 \\ +S_2a_2 \end{matrix} \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Forward Instantaneous Kinematics

- **Calculation of Jacobian**

Example (cont'd):

Recall that if joint *i* under consideration is a revolute joint, then:

$$^{T_6}J_{1i} = (-n_x p_y + n_y p_x) \quad ^{T_6}J_{2i} = (-o_x p_y + o_y p_x) \quad ^{T_6}J_{3i} = (-a_x p_y + a_y p_x)$$
$$^{T_6}J_{4i} = n_z \quad ^{T_6}J_{5i} = o_z \quad ^{T_6}J_{6i} = a_z$$

Then:

$$\begin{aligned} ^{T_6}J_{11} &= (-n_x p_y + n_y p_x) \\ &= -[C_1(C_{234}C_5C_6 - S_{234}S_6) - S_1S_5C_6] \times [S_1(C_{234}a_4 + C_{23}a_3 + C_2a_2)] \\ &\quad + [S_1(C_{234}C_5C_6 - S_{234}S_6) + C_1S_5C_6] \times [C_1(C_{234}a_4 + C_{23}a_3 + C_2a_2)] \\ &= S_5C_6(C_{234}a_4 + C_{23}a_3 + C_2a_2) \end{aligned}$$

$$^{T_6}J_{41} = n_z = S_{234}C_5C_6 + C_{234}S_6$$

Forward Instantaneous Kinematics

- **Calculation of Jacobian**

Example (cont'd):

$$J_{11} = -S_1\left[C_{234}a_4 + C_{23}a_3 + C_2a_2\right]$$

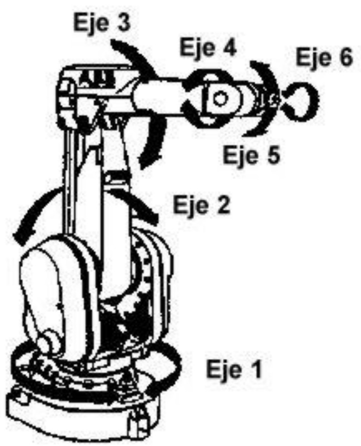
$$^{T_6}J_{11} = S_5C_6\left[C_{234}a_4 + C_{23}a_3 + C_2a_2\right]$$

As you can see, J_{11} are different. This is because one is relative to the reference frame, and the other is relative to the current or T_6 frame.

Outline

- Differential Kinematics
- Differential Motions
- Differential Motions of a Frame
- Forward Instantaneous Kinematics
- **Inverse Instantaneous Kinematics**
- Kinematic Singularity
- Summary

Inverse Instantaneous Kinematics

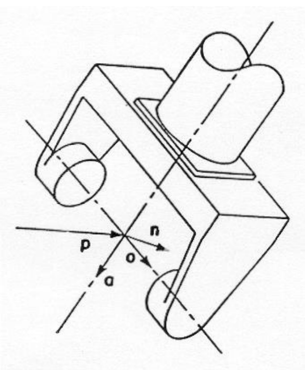


Serial chain manipulator

$$\begin{bmatrix} d\theta_1 \\ d\theta_2 \\ d\theta_3 \\ d\theta_4 \\ d\theta_5 \\ d\theta_6 \end{bmatrix} \longleftarrow \text{Inverse Instantaneous Kinematics} \quad \begin{bmatrix} dx \\ dy \\ dz \\ \delta x \\ \delta y \\ \delta z \end{bmatrix}$$

Given: The positions of all members of the chain and the total velocity of the end-effector.

Required: The rates of motion of all joints.



End-effector

Inverse Instantaneous Kinematics

In order to calculate the differential motions (or velocities) needed at the joints of the robot for a desired hand differential motion (or velocity), we need to calculate the **inverse of the Jacobian** and use it in the following equation:

$$[D] = [J][D_\theta]$$

$$[J^{-1}][D] = [J^{-1}][J][D_\theta] \rightarrow [D_\theta] = [J^{-1}][D]$$

and similarly:

$$\left[{}^{T_6}J^{-1}\right]\left[{}^{T_6}D\right] = \left[{}^{T_6}J^{-1}\right]\left[{}^{T_6}J\right][D_\theta] \rightarrow D_\theta = \left[{}^{T_6}J^{-1}\right]\left[{}^{T_6}D\right]$$

This means that knowing the **inverse of the Jacobian**, we can calculate how fast each joint must move, such that the robot's hand will yield a desired differential motion or velocity.

Inverse Instantaneous Kinematics

- **Inverting the Jacobian**

Inverting the Jacobian may be done in two ways; both are **very difficult, computationally intensive, and time consuming**.

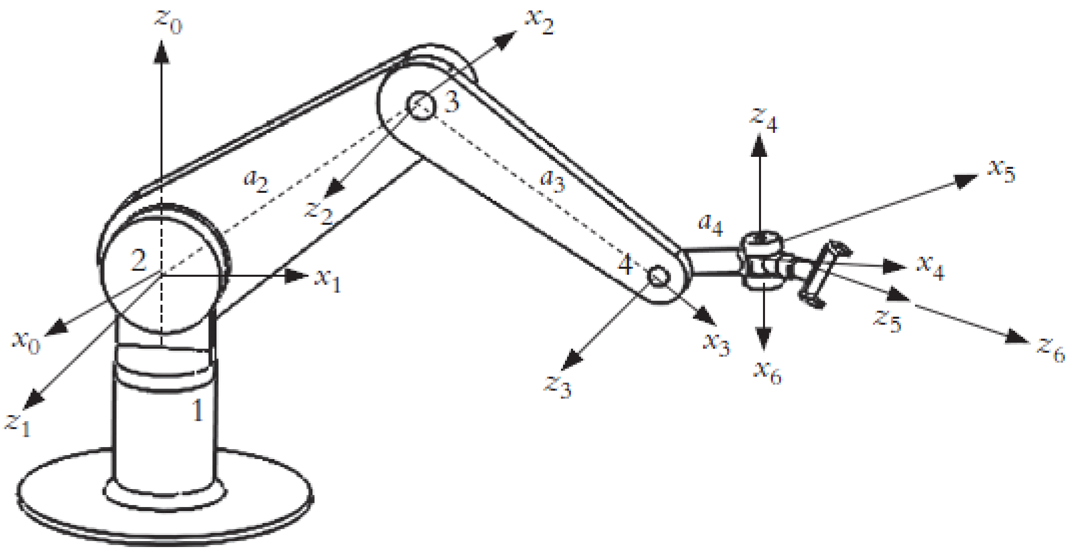
1. **Symbolic technique:** is to find the symbolic inverse of the Jacobian and then substitute the values into it to compute the velocities.
2. **Numerical technique:** second technique is to substitute the numbers in the Jacobian and then invert the numerical matrix through techniques such as Gaussian elimination or other similar approaches.

Although these are both possible, they are usually not done.

Inverse Instantaneous Kinematics

- **Inverting the Jacobian**

  Instead, we may use the inverse kinematic equations to calculate the joint velocities.



$$p_x S_1 - p_y C_1 = 0 \;\rightarrow\; \theta_1 = \tan^{-1}\left(\frac{p_y}{p_x}\right)$$

Inverse Instantaneous Kinematics

- **Inverting the Jacobian**

$$p_x S_1 - p_y C_1 = 0 \rightarrow \theta_1 = \tan^{-1}\left(\frac{p_y}{p_x}\right)$$

We can **differentiate the relationship to find $d\theta_1$**, which is the differential value of θ_1, as:

$$p_x S_1 = p_y C_1$$

$$dp_x S_1 + p_x C_1 d\theta_1 = dp_y C_1 - p_y S_1 d\theta_1$$

$$d\theta_1 (p_x C_1 + p_y S_1) = -dp_x S_1 + dp_y C_1$$

$$d\theta_1 = \frac{-dp_x S_1 + dp_y C_1}{(p_x C_1 + p_y S_1)}$$

Similarly, you can calculate the differential motions of the other joints...

Inverse Instantaneous Kinematics

- **Inverting the Jacobian**

Example: Given the following:

$$D = \begin{bmatrix} 0.01 \\ 0.02 \\ 0.03 \end{bmatrix} \quad J = \begin{bmatrix} 5 & 10 & 0 \\ 3 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

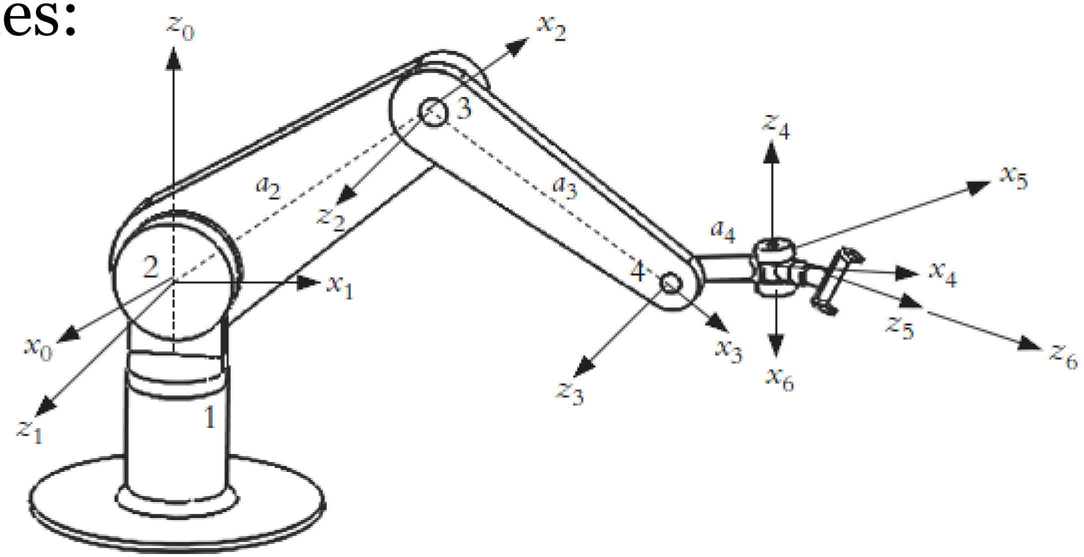
Find the values of joint differential motions for the three joints (we will call them ds_1, $d\theta_2$, $d\theta_3$) of the robot that caused the given frame change.

Solution:

$$D_\theta = J^{-1} \cdot D = \begin{bmatrix} 0 & 0.333 & 0 \\ 0.1 & -0.167 & 0 \\ -0.1 & 0.167 & 1 \end{bmatrix} \times \begin{bmatrix} 0.01 \\ 0.02 \\ 0.03 \end{bmatrix} = \begin{bmatrix} 0.0067 \\ -0.0023 \\ 0.0323 \end{bmatrix}$$

Inverse Instantaneous Kinematics

Example: The revolute robot is in the following configuration. Calculate the angular velocity of the first joint for the given values such that the hand frame will have the following linear and angular velocities:

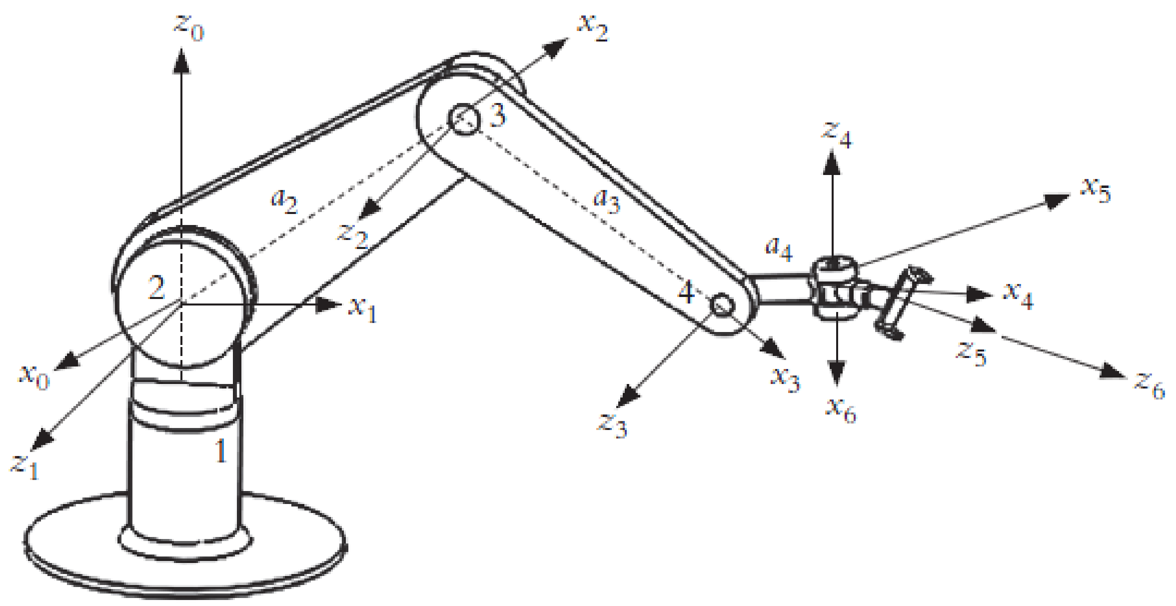


$$dx/dt = 1 \text{ in/sec} \quad dy/dt = -2 \text{ in/sec} \quad \delta x/dt = 0.1 \text{ rad/sec}$$

$$\theta_1 = 0°, \quad \theta_2 = 90°, \quad \theta_3 = 0°, \quad \theta_4 = 90°, \quad \theta_5 = 0°, \quad \theta_6 = 45°$$
$$a_2 = 15'', \quad a_3 = 15'', \quad a_4 = 5''$$

Inverse Instantaneous Kinematics

Given:



$$dx/dt = 1 \text{ in/sec} \quad dy/dt = -2 \text{ in/sec} \quad \delta x/dt = 0.1 \text{ rad/sec}$$

$$\theta_1 = 0°, \quad \theta_2 = 90°, \quad \theta_3 = 0°, \quad \theta_4 = 90°, \quad \theta_5 = 0°, \quad \theta_6 = 45°$$
$$a_2 = 15'', \quad a_3 = 15'', \quad a_4 = 5''$$

Required: $\frac{d\theta_1}{dt}$

Inverse Instantaneous Kinematics

Solution:

$$dx/dt = 1 \text{ in/sec} \quad dy/dt = -2 \text{ in/sec} \quad \delta x/dt = 0.1 \text{ rad/sec}$$

$$\theta_1 = 0°, \quad \theta_2 = 90°, \quad \theta_3 = 0°, \quad \theta_4 = 90°, \quad \theta_5 = 0°, \quad \theta_6 = 45°$$
$$a_2 = 15'', \quad a_3 = 15'', \quad a_4 = 5''$$

From **inverse instantaneous kinematics** solution:

$$d\theta_1 = \frac{-dp_x S_1 + dp_y C_1}{(p_x C_1 + p_y S_1)}$$

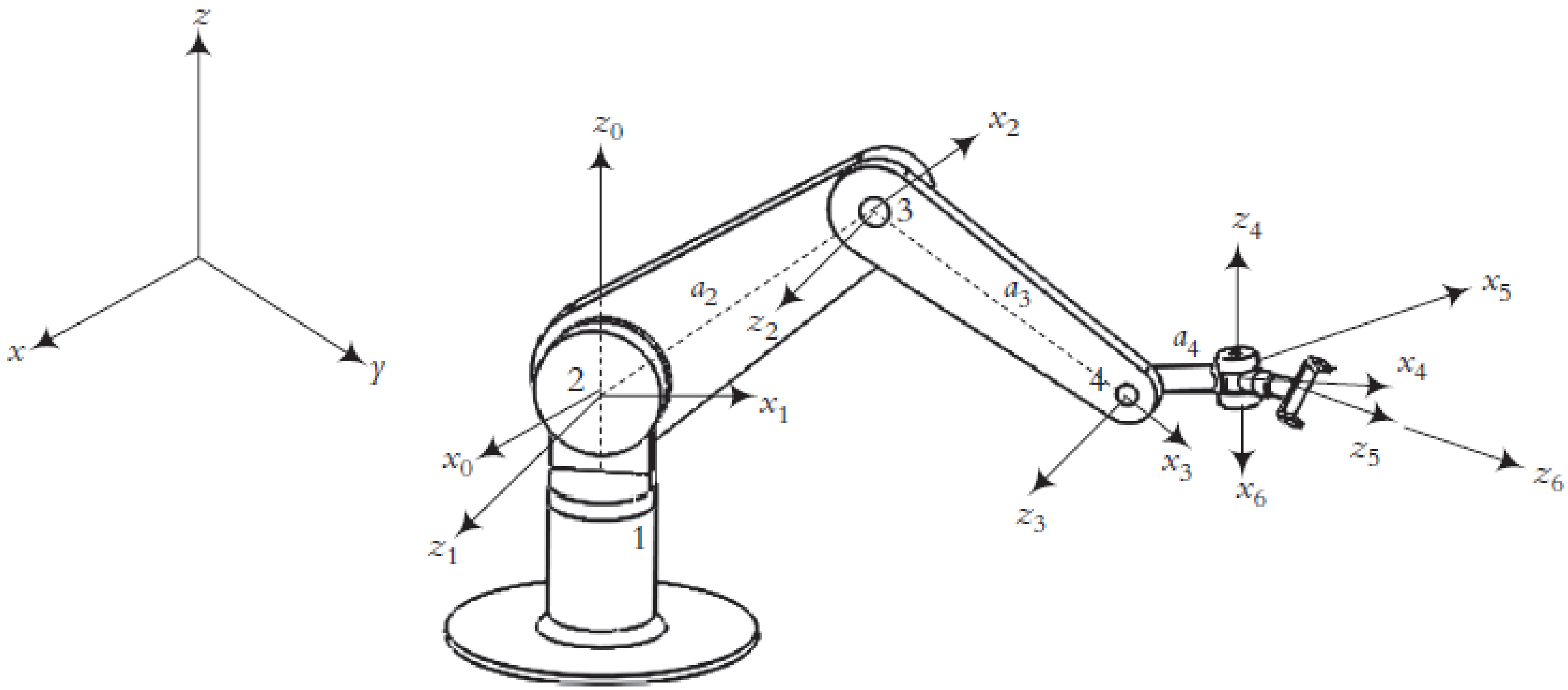
How to calculate dp_x, dp_y, p_x and p_y ?

$$^{R}T_H = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \Delta = \begin{bmatrix} 0 & -\delta z & \delta y & dx \\ \delta z & 0 & -\delta x & dy \\ -\delta y & \delta x & 0 & dz \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Inverse Instantaneous Kinematics

Solution:

#	θ	d	a	α
1	θ_1	0	0	90
2	θ_2	0	a_2	0
3	θ_3	0	a_3	0
4	θ_4	0	a_4	−90
5	θ_5	0	0	90
6	θ_6	0	0	0



Inverse Instantaneous Kinematics

Solution:

$$
{}^{R}T_{H} = A_1A_2A_3A_4A_5A_6
$$

$$
= \begin{bmatrix} C_1(C_{234}C_5C_6 - S_{234}S_6) & C_1(-C_{234}C_5C_6 - S_{234}C_6) & C_1(C_{234}S_5) & \\ -S_1S_5C_6 & +S_1S_5S_6 & +S_1C_5 & C_1(C_{234}a_4 + C_{23}a_3 + C_2a_2) \\ S_1(C_{234}C_5C_6 - S_{234}S_6) & S_1(-C_{234}C_5C_6 - S_{234}C_6) & S_1(C_{234}S_5) & \\ +C_1S_5C_6 & -C_1S_5S_6 & -C_1C_5 & S_1(C_{234}a_4 + C_{23}a_3 + C_2a_2) \\ S_{234}C_5C_6 + C_{234}S_6 & -S_{234}C_5C_6 + C_{234}C_6 & S_{234}S_5 & S_{234}a_4 + S_{23}a_3 + S_2a_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{R}T_{H} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.707 & 0.707 & 0 & -5 \\ 0 & 0 & -1 & 0 \\ -0.707 & -0.707 & 0 & 30 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Inverse Instantaneous Kinematics

Solution:

Substituting the desired differential motion values

$$\Delta = \begin{bmatrix} 0 & -\delta z & \delta y & dx \\ \delta z & 0 & -\delta x & dy \\ -\delta y & \delta x & 0 & dz \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -0.1 & -2 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[dT] = [\Delta][T] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0.0707 & 0.0707 & 0 & -5 \\ 0 & 0 & -0.1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Inverse Instantaneous Kinematics

Solution:

Since $$d\theta_1 = \frac{-dp_x S_1 + dp_y C_1}{(p_x C_1 + p_y S_1)}$$ and

$$^{R}T_H = \begin{bmatrix} -0.707 & 0.707 & 0 & -5 \\ 0 & 0 & -1 & 0 \\ -0.707 & -0.707 & 0 & 30 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad [dT] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0.0707 & 0.0707 & 0 & -5 \\ 0 & 0 & -0.1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Substituting the values from the **dT** and **T** matrices into the above equation, we get:

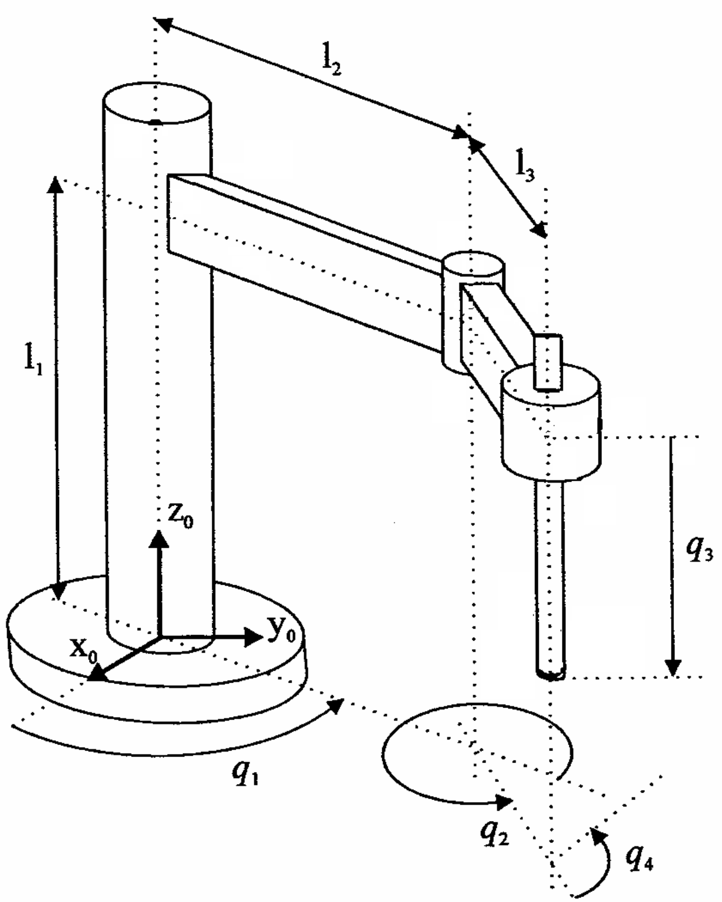
$$\frac{d\theta_1}{dt} = \frac{-dp_x S_1 + dp_y C_1}{(p_x C_1 + p_y S_1)} = \frac{-1(0) - 5(1)}{-5(1) + 0(0)} = 1 \text{ rad/sec}$$

Outline

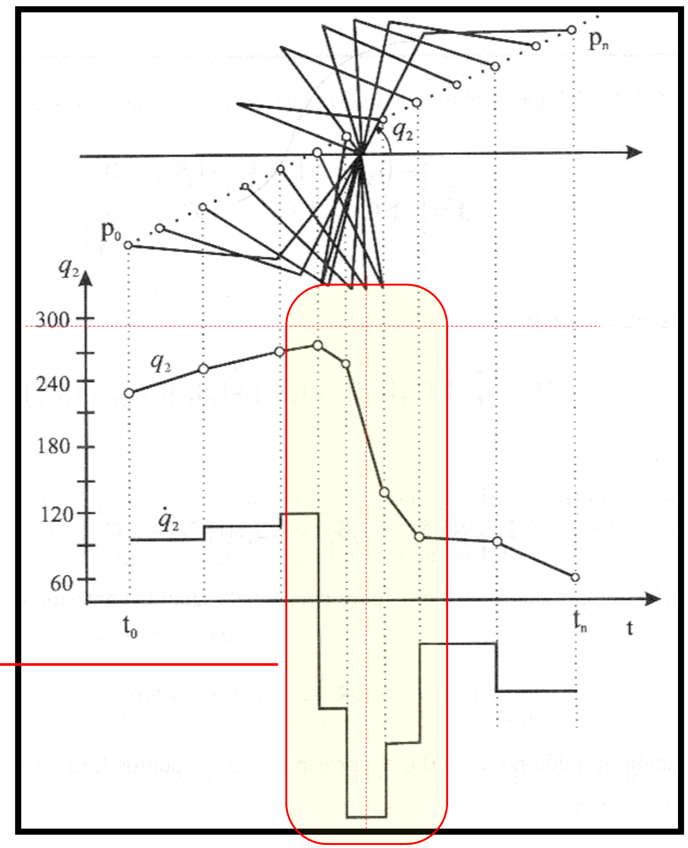
- Differential Kinematics
- Differential Motions
- Differential Motions of a Frame
- Forward Instantaneous Kinematics
- Inverse Instantaneous Kinematics
- **Kinematic Singularity**
- Summary

Kinematic Singularity

Singularity is the position or configuration of the manipulator where the subsequent behavior cannot be predicted, or a joint velocity become infinite or undeterministic.



Singular configuration



Kinematic Singularity

- **Why identifying manipulator singularities is important?**

1. Singularities represent configurations from which certain directions of motion may be unattainable.
2. At singularities, bounded end-effector velocities may correspond to unbounded joint velocities.
3. At singularities, bounded end-effector forces and torques may correspond to unbounded joint torques.
4. Singularities usually (but not always) correspond to points on the boundary of the manipulator workspace, that is, to points of maximum reach of the manipulator.

Kinematic Singularity

- **Why identifying manipulator singularities is important?**

5. Singularities correspond to points in the manipulator workspace that may be unreachable under small perturbations of the link parameters, such as length, offset, etc.
6. Near singularities there will not exist a unique solution to the inverse kinematics problem. In such cases there may be no solution or there may be infinitely many solutions.

Kinematic Singularity

At the singular points, the determinant of the Jacobian matrix is null.

{det [J]=0}

Example: Consider a 2-DOF planar arm represented by:

$$[\mathbf{D}] = \mathbf{J}[\mathbf{D}_\theta] \quad \text{or} \quad \begin{bmatrix} \mathbf{dx} \\ \mathbf{dy} \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{d\theta_1} \\ \mathbf{d\theta_2} \end{bmatrix}$$

that corresponds to the two equations

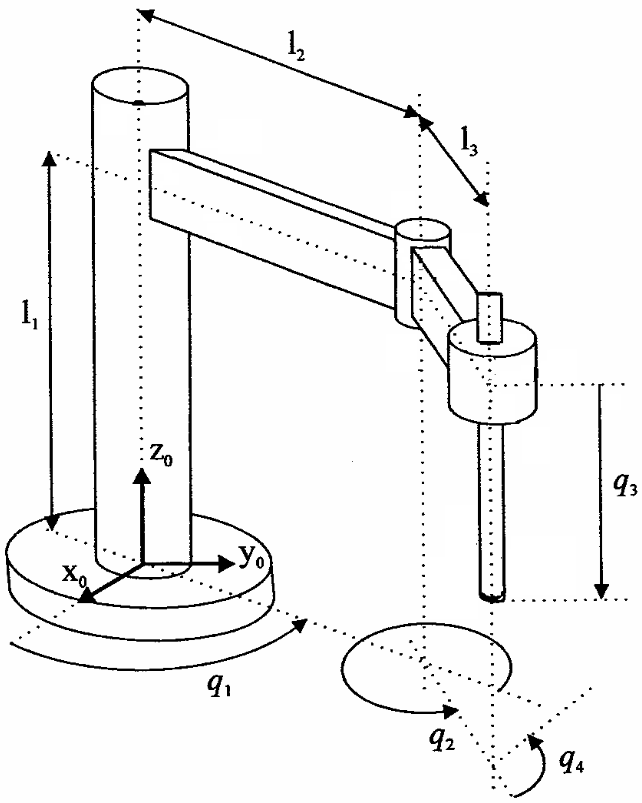
$$dx = d\theta_1 + d\theta_1$$
$$dy = 0$$

In this case the **det[J]=0**, and we see that for any values of the variables $d\theta_1$ and $d\theta_2$ there is no change in the variable dy. Thus any vector [D] having a nonzero second component represents an **unattainable direction** of instantaneous motion.

Kinematic Singularity

- **SCARA Robot**

$$\mathbf{J} = \begin{bmatrix} -(l_3 S_{12} + l_2 S_1) & -l_3 S_{12} & 0 \\ l_3 C_{12} + l_2 C_1 & l_3 C_{12} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$



The Jacobian is:

$$|J| = -[-l_3 C_{12}(l_3 S_{12} + l_2 S_1) + l_3 S_{12}(l_3 C_{12} + l_2 C_1)]$$

At the singular points

{det [J]=0}

$$l_3 C_{12}(l_3 S_{12} + l_2 S_1) = l_3 S_{12}(l_3 C_{12} + l_2 C_1)$$

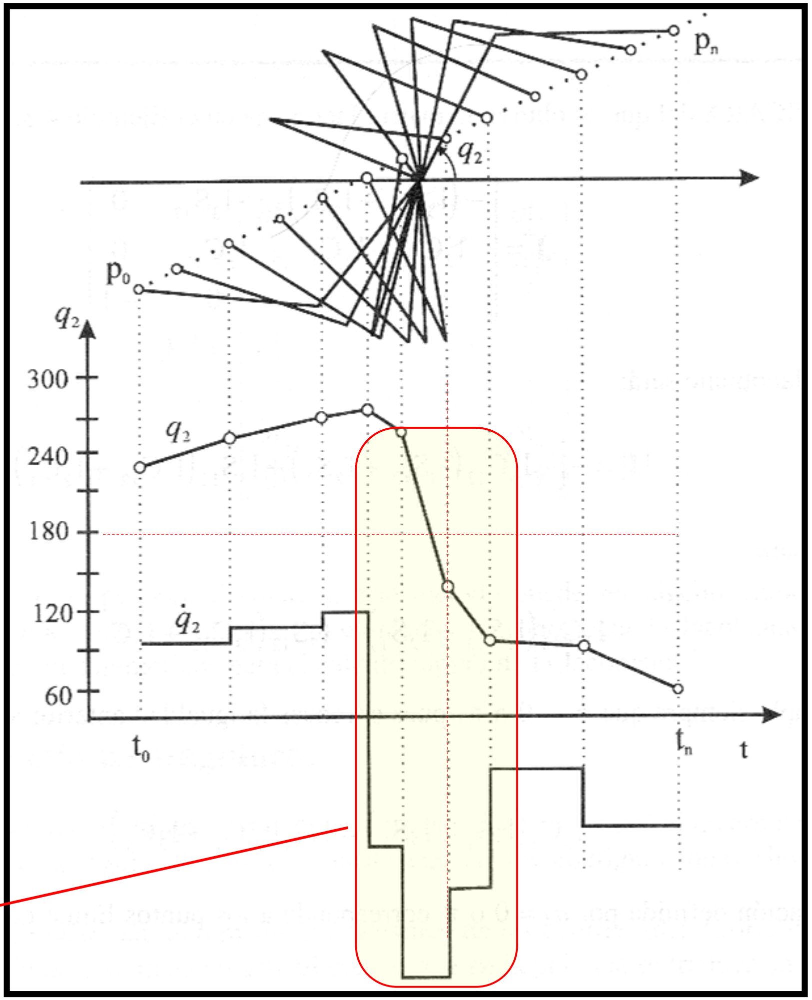
Kinematic Singularity

- **SCARA Robot (cont'd)**

$$l_3 C_{12}(l_3 S_{12} + l_2 S_1) = l_3 S_{12}(l_3 C_{12} + l_2 C_1)$$

These can be achieved when q_2=0 or π

- q_2=0: Outer limit of the work space
- q_2=π: Inner limit of the work space



Abrupt change in the velocity

Outline

- Differential Kinematics
- Differential Motions
- Differential Motions of a Frame
- Forward Instantaneous Kinematics
- Inverse Instantaneous Kinematics
- Kinematic Singularity
- **Summary**

Summary

- Forward instantaneous kinematics calculates how fast a robot's hand moves in space if the joint velocities are known.
- Inverse differential motion determines how fast each joint of a robot must move in order to generate a desired hand velocity.
- Together with the inverse kinematic equations of motion, we can control both the motions and the velocity of a multi-DOF robot in space. We can also follow the location of the hand frame as it moves in space.
- Kinematic singularity is the position or configuration of the manipulator where the subsequent behavior cannot be predicted, or a joint velocity become infinite or undeterministic.
- At a singular configuration it is impossible to generate end-effector task velocities or accelerations in certain directions.

Summary

- Given that any point of the workspace boundary represents a positioning singularity – different from an orientation singularity – manipulators with workspace boundaries that are not manifolds exhibit double singularities at the edges of their workspace boundary, which means that at edge points the rank of the robot Jacobian becomes deficient by two. At any other point of the workspace boundary the rank deficiency is by one.